

# Colorings with Fractional Defect

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## Abstract

Consider a coloring of a graph such that each vertex is assigned a fraction of each color, with the total amount of colors at each vertex summing to 1. We define the fractional defect of a vertex  $v$  to be the sum of the overlaps with each neighbor of  $v$ , and the fractional defect of the graph to be the maximum of the defects over all vertices. Note that this coincides with the usual definition of defect if every vertex is monochromatic. We provide results on the minimum fractional defect of 2-colorings of some graphs.

## 1 Introduction

In a usual vertex coloring of a graph, every vertex is assigned one color and that color is different from each of its neighbors. We consider here a two-fold generalization of this: a vertex can receive multiple colors and can overlap slightly with each neighbor.

Specifically, each vertex is assigned a fraction of each color, with the total amount of colors at each vertex summing to 1. The **(fractional) defect** of a vertex  $v$  is defined to be the sum of the overlaps over all colors and all neighbors of  $v$ . For example, if vertex  $v$  receives  $\frac{1}{3}$  red and  $\frac{2}{3}$  blue, while its neighbor  $w$  receives  $\frac{2}{5}$  red,  $\frac{2}{5}$  blue, and  $\frac{1}{5}$  white, then  $w$  contributes  $\frac{1}{3} + \frac{2}{5}$  to the defect of  $v$ , and vice versa. We define the **(fractional) defect** of the graph as the maximum of the defects over all vertices. Note that if every vertex is monochromatic (has only one color), then our fractional defect coincides with the usual definition of defect (see for example [2]); and that defective colorings are also called improper colorings.

The idea of assigning vertices multiple colors has been used most notably in fractional colorings (e.g. [8, 5]), but also for example in  $t$ -tone colorings [4]. Like in  $t$ -tone colorings (and unlike in fractional colorings), we consider here the situation where one pays for

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each color used, regardless of how much the color is used. Note that for proper colorings, allowing one to color a vertex with multiple colors does not yield anything new. For, one can just choose for each vertex  $v$  one color present at  $v$  and recolor it entirely that color, and therefore the minimum number of colors needed is just the chromatic number. Similarly, with the usual definition of the defect of a vertex as the number of neighbors that share a color, there is no advantage to using more than one color at a vertex. But we consider colorings where a vertex overlaps only slightly with each neighbor.

Consider, for example, the Hajós graph. Figure 1 gives a 2-coloring of this graph with defect  $4/3$  (and this is best possible in that any 2-coloring has at least this much defect). For another example, consider the complete graph on 3 vertices. Any 2-coloring of  $K_3$  has defect at least 1, but there are multiple optimal colorings: color one vertex red, one vertex blue, and the third vertex any combination of red and blue.

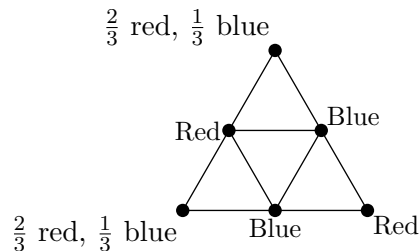


Figure 1: An optimal 2-coloring of the Hajós graph

Our objective is to minimize the defect of the graph. Specifically, for a given number of colors, what is the minimum defect that can be obtained? If the number of colors is the chromatic number, then of course the defect can be zero. But if the number of colors is smaller, then there is a positive defect.

In the rest of the paper we proceed as follows. In Section 2 we introduce some notation and prove a few general results. Thereafter in Section 3 we consider calculating the optimal 2-colorings for several graph families, including fans, wheels, complete multipartite graphs, rooks graphs, and regular graphs. We give exact values in some cases and bounds in others. We also pose several conjectures. Finally in Section 4 we observe that the decision problem is NP-hard.

## 2 Preliminaries

Consider a graph  $G$  and  $k$  a number of colors. For color  $j$ , let  $f_j(v)$  be the usage of color  $j$  on vertex  $v$ . Then the **defect** of vertex  $v$  is given by

$$df(v) = \sum_{w \in N(v)} \sum_{j=1}^k \min(f_j(v), f_j(w)). \quad (1)$$

In general, the problem is to minimize

$$\max_v df(v)$$

over all colorings such that  $f_j(v)$  is nonnegative and  $\sum_{j=1}^k f_j(v) = 1$  for all vertices  $v$ . We denote this minimum by  $D(G, k)$ , and call it the **minimum defect**.

Note that the existence of the minimum is guaranteed, since the objective function is continuous and the feasible region is a closed bounded set. Further, the calculation is at least finite, since, for example, we can prescribe which of  $f_j(v)$  and  $f_j(w)$  are smaller in every min in Equation 1 for each vertex  $v$ , and thus  $D(G, k)$  is the minimum over exponentially many linear programs.

A related parameter is the **minimum total defect**: we define  $TD(G, k)$  to be the minimum over all colorings of  $\sum_{v \in V} df(v)$ . However, here fractional colorings do not yield anything new:

**Lemma 1** *For graph  $G$  and number of colors  $k$ , there is a  $k$ -coloring that achieves  $TD(G, k)$  in which every vertex is monochromatic.*

**Proof.** Consider any vertex  $v$  that is not monochromatic: say  $f_1(v), f_2(v) > 0$  with  $f_1(v) + f_2(v) = A$ . Then consider adjusting the coloring such that  $f_1(v) = x$  and  $f_2(v) = A - x$ . As a function of  $x$ , the defect of  $v$  with any neighbor  $w$  is a (piecewise-linear) concave-down function. Thus, the total defect of the graph, as a function of  $x$ , is a concave-down function, and so its minimum is attained at an endpoint. This means that one can either replace color 1 by color 2 or replace color 2 by color 1 at  $v$  without increasing the total defect. Repeated application of this replacement yields a coloring with every vertex monochromatic.  $\square$

For example, we will often use the fact that:

**Corollary 1**  $TD(K_n, k) = \lfloor n/k \rfloor (2n - k - \lfloor n/k \rfloor k)$ .

There are several fundamental results about monochromatic vertices for minimum defect. One is that we may assume that there is a monochromatic vertex of each color.

**Lemma 2** *Let  $k$  be an integer and  $G$  be a graph with at least  $k$  vertices. Then there is a  $k$ -coloring that achieves  $D(G, k)$  that has at least one monochromatic vertex for each color.*

**Proof.** Consider any optimal  $k$ -coloring, and consider each color  $j = 1, 2, \dots, k$  in turn. Each time, define vertex  $v_j$  as any vertex other than  $v_1, \dots, v_{j-1}$  with the largest usage of color  $j$ ; then recolor  $v_j$  (if needed) such that  $f_j(v_j) = 1$  and  $f_i(v_j) = 0$  for all  $i \neq j$ . Such a recoloring does not increase the defect at any vertex. So we will reach an optimal coloring with the desired property.  $\square$

We next show that the minimum defect is either 0 or at least 1.

**Lemma 3** *For any graph  $G$  and positive integer  $k$ , if  $D(G, k) > 0$  then  $D(G, k) \geq 1$ .*

**Proof.** If every vertex is monochromatic, then the defect is an integer and so the result is immediate. So consider any vertex  $v$  that is not monochromatic. If for any color  $j$  we have  $f_j(v) \geq f_j(w)$  for all neighbors  $w$  of  $v$ , then we can recolor  $v$  to be monochromatically color  $j$  without increasing the defect of any vertex. So we may assume that for every color  $j$  at  $v$ , vertex  $v$  has a neighbor  $w_j$  with  $f_j(w_j) \geq f_j(v)$ . It follows that

$$df(v) = \sum_{w \in N(v)} \sum_{j=1}^k \min(f_j(v), f_j(w)) \geq \sum_{j=1}^k \min(f_j(v), f_j(w_j)) = \sum_{j=1}^k f_j(v) = 1.$$

The result follows.  $\square$

For example, it follows from Lemma 3 that the optimal 2-coloring of the odd cycle  $C_n$  has defect 1.

**Proposition 1** *The complete graph  $K_n$  has  $D(K_n, k) = \lceil n/k \rceil - 1$ .*

**Proof.** This defect is achieved by (inter alia) coloring each vertex with a single color and using each color as equitably as possible. (This is trivially the best coloring for total defect.)

To see that  $\lceil n/k \rceil - 1$  is best possible, we proceed by induction on  $n$ , noting that the result is trivial if  $n \leq k$ . So assume  $n > k$ . By Lemma 2, there is an optimal  $k$ -coloring with for each  $j$  a vertex  $v_j$  that is monochromatically color  $j$ . Let  $A = \{v_1, \dots, v_k\}$ . Then the defect of any other vertex  $w$  in  $G$  equals 1 plus the defect of  $w$  in  $G - A$ . By the induction hypothesis, there exists a vertex in  $G - A$  that has defect at least  $\lceil (n - k)/k \rceil - 1$  in  $G - A$ . This proves the lower bound.  $\square$

### 3 Two-Colorings of Some Graph Families

We now consider results for 2-colorings. Unless otherwise specified, we assume the colors are red and blue, and denote the red usage at vertex  $v$  by  $r(v)$  (so that the blue usage is  $1 - r(v)$ ). Unfortunately we are only able to provide results for very specific families of graphs.

#### 3.1 Fans

The **fan**, denoted by  $F_n$ , is the graph obtained from a path of order  $n$  by adding a new vertex and joining it to every vertex of the path.

**Lemma 4** *In any 2-coloring of  $F_3$  it holds that  $df(v) + df(w) \geq 2$  where  $v$  and  $w$  are the dominating vertices.*

**Proof.** Suppose the dominating vertices are  $v$  and  $w$  and the other two vertices are  $a$  and  $b$ . Let  $e_{xy}$  denote the overlap  $\min(r(x), r(y)) + \min(1 - r(x), 1 - r(y))$  between vertices  $x$  and  $y$ . Then  $df(v) + df(w) = e_{va} + e_{vb} + e_{wa} + e_{wb} + 2e_{vw}$ ; further, because a triangle has total defect at least 2 (Corollary 1), we have  $e_{va} + e_{wa} + e_{vw} \geq 1$  and  $e_{vb} + e_{wb} + e_{vw} \geq 1$ . The result follows.  $\square$

Note that  $F_1$  is just  $K_2$  and  $F_2$  is just  $K_3$ , and so it holds that  $D(F_1, 2) = 0$  and  $D(F_2, 2) = 1$ . For the general cases of  $F_n$ , we have the following:

**Proposition 2** *The minimum defect in a 2-coloring of  $F_n$  ( $n \geq 3$ ) is*

$$D(F_n, 2) = \frac{2\lfloor n/3 \rfloor}{\lfloor n/3 \rfloor + 1}.$$

**Proof.** Say the path is  $v_1v_2 \dots v_n$  with dominating vertex  $h$ .

We prove the upper bound by the following construction. Set  $x = 2/(\lfloor n/3 \rfloor + 1)$ . Let  $r(h) = 1$ . Let  $r(v_i) = x$  if  $i$  is a multiple of 3, and 0 otherwise. It can readily be checked that every vertex  $v_i$  has defect at most  $2 - x$ , and that vertex  $h$  has defect  $\lfloor n/3 \rfloor x$ . The result follows since both these values equal the claimed upper bound.

To prove the lower bound, it suffices to show that  $D(F_n, 2) \geq 2n/(n+3)$  if  $n$  is a multiple of 3. We partition the path  $P_n$  into  $n/3$  copies of  $P_3$ ; thus each  $P_3$  along with vertex  $h$  forms a copy of  $F_3$ . Note that for each  $1 \leq i \leq n/3$ , vertices  $h$  and  $v_{3i-1}$  are the dominating vertices of the  $i^{\text{th}}$  copy of  $F_3$ . It follows from Lemma 4 that each copy of  $F_3$  contributes at least 2 to the sum of the defects of  $h$  and  $v_{3i-1}$ . Therefore,  $df(h) + \sum_{i=1}^{n/3} df(v_{3i-1}) \geq 2n/3$ , whence the result.  $\square$

Note that the defect  $D(F_n, 2)$  tends to 2 as  $n$  increases. The fan is outerplanar. Several researchers [1, 7] showed that one can ordinarily 2-color an outerplanar graph with defect at most 2. However, we conjecture that this bound can be improved slightly in the following sense:

**Conjecture 1**  $D(G, 2) < 2$  for any outerplanar graph  $G$ .

### 3.2 Wheels

The **wheel**, denoted by  $W_n$ , is the graph formed from a cycle of order  $n$  by adding a new vertex and joining it to every vertex of the cycle. The vertex of degree  $n$  is called the **hub** of the wheel.

**Proposition 3** *For  $n \geq 3$ , the minimum defect in a 2-coloring of  $W_n$  is*

$$D(W_n, 2) = \frac{2\lceil n/3 \rceil}{\lceil n/3 \rceil + 1}.$$

**Proof.** Let  $x = 2/(\lceil n/3 \rceil + 1)$  and let  $D$  be a minimum independent dominating set of the cycle. For a vertex  $v$  on the cycle, let  $r(v) = x$  if  $v \in D$ , and  $r(v) = 0$  otherwise. Let  $r(h) = 1$  for the hub  $h$ . It can readily be checked that every vertex on the cycle has defect at most  $2 - x$ , and that the hub has defect  $x|D|$ . The upper bound follows, since  $2 - x = x|D| = 2\lceil n/3 \rceil / (\lceil n/3 \rceil + 1)$ .

Next we prove the lower bound. When  $n = 3k$ , the lower bound follows directly from Proposition 2. Indeed,  $D(W_{3k}, 2) \geq D(F_{3k}, 2) = 2k/(k + 1)$ . So we need to establish the lower bound for  $n = 3k + 1$  and  $n = 3k + 2$ .

Consider an optimal coloring of  $W_n$  with hub  $h$  and cycle  $v_1, v_2, \dots, v_n, v_1$ . By Lemma 2, we may assume there exist vertices  $u$  and  $u'$  such that  $r(u) = 0$  and  $r(u') = 1$ . There are two cases.

(a) Assume  $h \notin \{u, u'\}$ . Then we can form  $k - 1$  edge-disjoint copies of  $P_3$  without using vertex  $h$ ,  $u$ , or  $u'$ . Let  $S$  denote the set of centers of these copies. By Lemma 4, it follows that the total defect of  $S \cup \{h\}$  within these copies is at least  $2(k - 1)$ . Further, vertices  $u$  and  $u'$  together contribute defect 1 to the hub  $h$ . It follows that, in the graph as a whole,  $df(h) + \sum_{s \in S} df(s) \geq 2k - 1$ , and so  $G$  has defect at least  $(2k - 1)/k$ . If  $k \geq 2$ , then  $(2k - 1)/k \geq (2k + 2)/(k + 2)$ , and we are done.

So consider the case when  $k = 1$ . Assume first that  $n = 5$ . Suppose  $u$  and  $u'$  are consecutive on the cycle; say  $u = v_1$  and  $u' = v_2$ . Then  $G - \{u, u'\}$  is a copy of  $F_3$ . Since  $u$  and  $u'$  together contribute defect 1 to  $h$ , it follows from Lemma 4 that  $df(h) + df(v_4) \geq 3$ , and so  $G$  has defect at least  $3/2$ . So assume without loss of generality that  $u = v_1$  and  $u' = v_3$ . If the hub has both two neighbors at least as red and two neighbors at most as red, then it has defect at least 2. So without loss of generality, we may assume that  $r(v_2), r(v_4), r(v_5) \leq r(h)$ . Then, the defect that  $h$  receives from  $\{v_2, v_5\}$  is  $2 - 2r(h) + r(v_2) + r(v_5)$ , and the defect that  $u$  receives from  $\{v_2, v_5\}$  is  $2 - r(v_2) - r(v_5)$ . That is, the sum of the defects that  $h$  and  $u$  receive from  $\{v_2, v_5\}$  is at least 2. Since  $h$  also receives defect 1 from  $u$  and  $u'$ , it follows that  $df(u) + df(h) \geq 3$ , and the result follows. The argument for  $n = 4$  is similar and omitted.

(b) Assume  $h \in \{u, u'\}$ . Say  $u = v_1$  and  $u' = h$ . Consider  $v_n$ . It receives defect 1 from  $\{v_1, h\}$ . Let index  $j$  be such that  $v_j$  is the redder vertex of  $v_{n-1}$  and  $v_n$ . Then  $v_{n-1}$  and  $v_n$  have at least  $1 - r(v_j)$  of blue overlap and so  $df(v_n) \geq 2 - r(v_j)$ .

Further, one can form  $k$  edge-disjoint copies of  $P_3$  without using vertex  $h$  or  $v_j$ . Let  $S$  denote the set of centers of these copies. By Lemma 4 and noting that the hub  $h$  receives defect  $r(v_j)$  from vertex  $v_j$ , it follows that  $df(h) + \sum_{s \in S} df(s) \geq 2k + r(v_j)$ .

Thus  $df(h) + df(v_n) + \sum_{s \in S} df(s) \geq 2k + 2$ , whence the result.  $\square$

### 3.3 Complete multipartite graphs and compositions

We consider here complete multipartite graphs. These can be thought of as taking a complete graph and replacing each vertex by an independent set with the same adjacency. In general, we define  $G[aK_1]$  to be the **composition** of  $G$  with the empty graph on  $a$  vertices; that is, the graph obtained by replacing every vertex  $v$  of  $G$  with a set  $I_v$  of size  $a$  such that a vertex of  $I_v$  is adjacent to a vertex of  $I_w$  if and only if  $v$  and  $w$  are adjacent in  $G$ .

There are two simple bounds:

**Proposition 4** *For any graph  $G$ ,*

- (a)  $TD(G[aK_1], k) \geq a^2 TD(G, k)$ .
- (b)  $D(G[aK_1], k) \leq aD(G, k)$ .

**Proof.** (a) The bound follows by applying the lower bound to the  $a^n$  copies of  $G$  and averaging.

(b) Take the optimal coloring of  $G$  and replicate it: give every vertex of  $I_v$  the color of vertex  $v$ .  $\square$

We let  $K_a^{(m)}$  denote the complete  $m$ -partite graph with  $a$  vertices in each partite set; that is  $K_a^{(m)} = K_m[aK_1]$ . It follows that:

**Proposition 5** *If  $m$  is a multiple of  $k$ , then the complete multipartite graph  $K_a^{(m)}$  can be  $k$ -colored with defect  $(m/k - 1)a$ , and this is best possible.*

But if  $m$  is not a multiple of  $k$ , the result is not clear. Perhaps the following is true.

**Conjecture 2** *The minimum defect in a  $k$ -coloring of  $K_a^{(m)}$  is  $(\lceil m/k \rceil - 1)a$ .*



In fact, we do not have an example that precludes it being the case that it always holds that  $D(G[aK_1], k) = aD(G, k)$ .

We shall prove Conjecture 2 for 2 colors. We need the following definitions. Define a vertex  $x$  as **large** if  $r(x) > 1/2$  and **small** if  $r(x) < 1/2$ . Also we let  $N(x)$  denote the set of neighbors of  $x$ ,  $U(x)$  denote the set of vertices  $y$  in  $N(x)$  with  $r(y) \geq r(x)$ , and  $L(x)$  denote the set of vertices  $y$  in  $N(x)$  with  $r(y) < r(x)$ .

We also need the following observations and lemmas. Some of them are very easy to verify and so the proofs are omitted:

**Observation 1** *If  $r(x) = 1/2$ , then  $df(x) \geq |N(x)|/2$ .*

**Observation 2** *If two vertices are both large (or both small), then the overlap between them is greater than  $1/2$ .*

**Observation 3**  $df(x) \geq \min(|U(x)|, |L(x)|)$ .

**Lemma 5**  $df(x) \geq |N(x)|/2$  if either

- (a)  $x$  is large and  $|U(x)| \geq |L(x)|$ ,
- or (b)  $x$  is small and  $|U(x)| \leq |L(x)|$ .

**Proof.** It suffices to prove it for the case that  $x$  is large. We pair each vertex in  $L(x)$  with a vertex in  $U(x)$ . Then each pair contributes at least 1 to  $df(x)$ . By Observation 2, each of the remaining vertices in  $U(x)$  contributes more than  $1/2$  to  $df(x)$ . Hence  $df(x) \geq |N(x)|/2$ .  $\square$

**Lemma 6** *If  $x$  is large and  $y$  is small, then  $\max(df(x), df(y)) \geq |N(x) \cap N(y)|/2$ .*

**Proof.** If  $|U(x)| \geq |L(x)|$ , then we have  $df(x) \geq |N(x)|/2 \geq |N(x) \cap N(y)|/2$  by Lemma 5. So we may assume  $|U(x)| < |L(x)|$ . Similarly we may assume  $|U(y)| > |L(y)|$ . Note that we can increase  $r(x)$  to 1 and decrease  $r(y)$  to 0 without increasing the defect of either vertex. It follows that  $df(x) + df(y)$  is at least their common degree, whence the result.  $\square$

**Lemma 7** *If all neighbors of  $x$  are large (small), then  $r(x)$  can be changed to 0 (1) without increasing the defect of any vertex.*

**Proof.** It suffices to prove it for the case that all neighbors of  $x$  are large. Let  $v$  be any neighbor of  $x$ . The overlap between them is  $1 - |r(v) - r(x)|$ . If  $r(x)$  is changed to 0, then the overlap becomes  $1 - r(v)$ . Since  $r(v) > 1/2$ , we have  $1 - |r(v) - r(x)| \geq 1 - r(v)$  and the conclusion follows.  $\square$

**Proposition 6** *The minimum defect in a 2-coloring of  $K_a^{(m)}$  is  $(\lceil m/2 \rceil - 1)a$ .*

**Proof.** Such defect is attained by coloring all vertices in  $\lfloor m/2 \rfloor$  of the partite sets with red, and the remaining vertices blue. So we need to prove that this is best possible.

If  $m$  is even, Proposition 4 shows that  $TD(K_a^{(m)}, 2) \geq m(m/2 - 1)a^2$ , and thus some vertex has defect at least  $(m/2 - 1)a$ . So assume  $m$  is odd.

If there is a vertex  $v$  in the graph with  $r(v) = 1/2$ , then  $df(v) \geq (m - 1)a/2 = (\lceil m/2 \rceil - 1)a$  by Observation 1. Also, if there is a partite set that contains both a large vertex and a small vertex, then we are okay by Observation 6.

Hence we may assume every partite set contains either only large vertices or only small vertices. Without loss of generality, assume at least  $(m + 1)/2$  partite sets contain only large vertices. Let  $x$  be the large vertex with *minimum*  $r(x)$ . Note that  $|U(x)| \geq (m - 1)a/2 \geq |L(x)|$ , and therefore  $df(x) \geq (m - 1)a/2 = (\lceil m/2 \rceil - 1)a$  by Observation 5.  $\square$

**Proposition 7** *The minimum defect in a 2-coloring of the complete tripartite graph  $K_{a,b,c}$  with  $a \leq b \leq c$  is  $bc/(b + c - a)$ .*

**Proof.** Let  $A$ ,  $B$ , and  $C$  denote the partite sets of order  $a$ ,  $b$ , and  $c$ , respectively. The upper bound is attained by coloring all vertices  $v$  in  $A$  with  $r(v) = 0$ , all vertices in  $C$  with  $r(v) = 1$ , and all vertices in  $B$  with  $r(v) = x$ , where  $x$  is chosen to give the vertices in  $A$  and  $B$  the same defect, namely  $x = (b - a)/(b + c - a)$ .

Now we prove the lower bound. Let  $x_1, x_2, \dots, x_a$  be the vertices in  $A$  with  $r(x_1) \leq r(x_2) \leq \dots \leq r(x_a)$ ,  $y_1, y_2, \dots, y_b$  be the vertices in  $B$  with  $r(y_1) \leq r(y_2) \leq \dots \leq r(y_b)$ ,

and  $z_1, z_2, \dots, z_c$  be the vertices in  $C$  with  $r(z_1) \leq r(z_2) \leq \dots \leq r(z_c)$ . There are two possible cases.

**Case 1:**  $a \leq b \leq c \leq a + b$ .

Then we have  $(b + c)/2 \geq (a + c)/2 \geq (a + b)/2 \geq bc/(b + c - a)$ . If there is a vertex  $v$  in the graph with  $r(v) = 1/2$ , then the conclusion follows from Observation 1. Also, if there is a partite set that contains both a large vertex and a small vertex, then the conclusion follows from Lemma 6. Hence we may assume that every partite set contains either only large vertices or only small vertices, and by symmetry we only need to consider the following four subcases:

**Case 1.1:** *all vertices in the graph are large.*

By Observation 2, we have  $df(x_i) \geq (b + c)/2$  for every  $1 \leq i \leq a$ . So the conclusion follows.

**Case 1.2:** *all vertices in  $A$  are small and all the other vertices are large.*

Let  $u$  be the large vertex with *minimum*  $r(u)$ . By Lemma 5,  $df(u) \geq (a + b)/2$  and the conclusion follows.

**Case 1.3:** *all vertices in  $B$  are small and all the other vertices are large.*

By Lemma 7, we may assume  $r(y_j) = 0$  for every  $1 \leq j \leq b$ . If  $r(x_1) \leq r(z_1)$ , then by Lemma 5,  $df(x_1) \geq (b + c)/2$ . So assume  $r(x_1) > r(z_1)$ . We have

$$\begin{aligned} df(x_a) &= b(1 - r(x_a)) + \sum_{k=1}^c (1 - |r(x_a) - r(z_k)|) \\ &\geq b(1 - r(x_a)) + \sum_{k=1}^c (r(x_a) + r(z_k) - 1) \\ &= (b - c)(1 - r(x_a)) + \sum_{k=1}^c r(z_k) \\ &\geq (b - c)(1 - r(x_a)) + c r(z_1), \end{aligned}$$

and

$$\begin{aligned} df(z_1) &= \sum_{i=1}^a (1 - (r(x_i) - r(z_1))) + b(1 - r(z_1)) \\ &= \sum_{i=1}^a (1 - r(x_i)) + (a - b)r(z_1) + b \\ &\geq a(1 - r(x_a)) + (a - b)r(z_1) + b. \end{aligned}$$

Hence,  $(b-a) df(x_a) + c df(z_1) \geq [(b-a)(b-c) + ac](1 - r(x_a)) + bc \geq bc$ . It follows that  $\max(df(x_a), df(z_1)) \geq bc/(b+c-a)$ .

**Case 1.4:** *all vertices in  $C$  are small and all the other vertices are large.*

By Lemma 7, we may assume  $r(z_k) = 0$  for every  $1 \leq k \leq c$ . If  $r(x_1) \leq r(y_1)$ , then we have

$$\begin{aligned} df(x_1) &= c(1 - r(x_1)) + \sum_{j=1}^b (1 - (r(y_j) - r(x_1))) \\ &= c + (b-c)r(x_1) + \sum_{j=1}^b (1 - r(y_j)) \\ &\geq c + (b-c)r(x_1), \end{aligned}$$

and

$$\begin{aligned} df(y_1) &= \sum_{i=1}^a (1 - |r(x_i) - r(y_1)|) + c(1 - r(y_1)) \\ &\geq \sum_{i=1}^a (r(x_i) + r(y_1) - 1) + c(1 - r(y_1)) \\ &= (c-a)(1 - r(y_1)) + \sum_{i=1}^a r(x_i) \\ &\geq \sum_{i=1}^a r(x_i) \\ &\geq a r(x_1). \end{aligned}$$

Hence, we have

$$\begin{aligned} b df(x_1) + (c-a) df(y_1) &\geq bc + [b(b-c) + (c-a)a]r(x_1) \\ &= bc + (b+a-c)(b-a)r(x_1) \\ &\geq bc. \end{aligned}$$

It follows that  $\max(df(x_1), df(y_1)) \geq bc/(b+c-a)$ .

Similarly, if  $r(x_1) > r(y_1)$ , then it can be verified that

$$df(x_1) \geq (c-b)(1 - x_1) + \sum_{j=1}^b r(y_j) \geq b r(y_1),$$

and

$$df(y_1) = \sum_{i=1}^a (1 - r(x_i)) + c + (a - c)r(y_1) \geq c + (a - c)r(y_1).$$

Hence, we have

$$(c - a) df(x_1) + b df(y_1) \geq b(c - a)r(y_1) + bc + b(a - c)r(y_1) = bc.$$

It follows that  $\max(df(x_1), df(y_1)) \geq bc/(b + c - a)$ .

**Case 2:**  $a \leq b < a + b < c$ .

Then we have  $(b + c)/2 \geq (a + c)/2 > c/2 > bc/(b + c - a)$ . By Observation 1 and Lemma 6, we only consider the case that the vertices of  $A \cup B$  are either all large or all small. Without loss of generality, assume that they are all large. Then by Lemma 7, we may assume that  $r(z_k) = 0$  for every  $1 \leq k \leq c$ .

If  $r(x_1) \leq r(y_1)$ , then  $df(x_1) \geq b$  by Observation 3. So assume  $r(x_1) > r(y_1)$ . But then by the same argument as that in Case 1.4, we have  $\max(df(x_1), df(y_1)) \geq bc/(b + c - a)$ .  $\square$

For another composition, consider  $C_m[aK_1]$  where  $m$  is odd. We now prove that  $D(C_m[2K_1], 2) = 2$ . There are at least two different optimal colorings. The first such coloring is obtained by taking an optimal coloring for  $C_m$  and replicating it. The second such coloring is obtained by, for each copy of  $2K_1$ , coloring one vertex red and one vertex blue.

**Proposition 8** *For  $m$  odd,  $D(C_m[2K_1], 2) = 2$ .*

**Proof.** Consider a 2-coloring of  $C_m[2K_1]$ . We need to show that the defect is at least 2. As in the proof of Proposition 6, we may assume that every copy of  $2K_1$  contains either two large vertices or two small vertices. Since  $m$  is odd, it follows that there must be two adjacent copies of the same type. Without loss of generality, assume  $u_1$  and  $u_2$  are adjacent to  $v_1$  and  $v_2$  with all four vertices being large. If any  $x \in \{u_1, u_2, v_1, v_2\}$  has  $|U(x)| \geq 2$ , then the lower bound follows from Lemma 5(a). Therefore we may assume that  $|U(x)| \leq 1$  for every  $x \in \{u_1, u_2, v_1, v_2\}$ . This means that each  $u_i$  is redder than some  $v_j$  and vice versa, a contradiction.  $\square$

### 3.4 Rooks graphs and Cartesian products

Recall that the **Cartesian product** of graphs  $G$  and  $H$ , denoted  $G \square H$ , is the graph whose vertex set is  $V(G) \times V(H)$ , in which two vertices  $(u_1, u_2)$  and  $(v_1, v_2)$  are adjacent if  $u_1 v_1 \in E(G)$  and  $u_2 = v_2$ , or  $u_1 = v_1$  and  $u_2 v_2 \in E(H)$ .

We will need the obvious lower bound for the total defect of Cartesian products.

**Proposition 9** *Let  $G$  and  $H$  be graphs of order  $m$  and  $n$  respectively. Then*  
 $TD(G \square H, k) \geq m TD(H, k) + n TD(G, k).$

**Proof.** The defect of a vertex in the product is the sum of the defects in its copies of  $G$  and  $H$ .  $\square$

The **rooks graph**, denoted by  $K_m \square K_n$ , is the Cartesian product of the complete graphs  $K_m$  and  $K_n$ . We will denote the vertices by  $(i, j)$  with  $1 \leq i \leq m, 1 \leq j \leq n$ .

**Lemma 8** *The rooks graph  $K_m \square K_n$  can be 2-colored with defect  $\lceil m/2 \rceil + \lceil n/2 \rceil - 2$ .*

**Proof.** Color vertex  $(i, j)$  with red if  $i$  and  $j$  have the same parity and blue otherwise.  $\square$

**Corollary 2** *Let  $m$  and  $n$  be even integers. Then  $D(K_m \square K_n, 2) = m/2 + n/2 - 2$ .*

**Proof.** The upper bound follows from Lemma 8. The lower bound follows from Proposition 9, since  $TD(K_s, 2) = s(s/2 - 1)$  for  $s$  even (Corollary 1), and thus  $TD(K_m \square K_n, 2) \geq mn(n/2 - 1) + nm(m/2 - 1)$ .  $\square$

We show below that the upper bound in Lemma 8 is not always optimal. In fact we conjecture that it is never optimal when  $m$  and  $n$  are both odd, except for the case that  $m = n = 3$ .

**Lemma 9**  $D(K_3 \square K_3, 2) = 2$ .

**Proof.** The upper bound is from Lemma 8.

We have two proofs of the lower bound, one by computer and one by hand. Both proofs entail converting the question to a set of linear programs.

Observe that given a coloring, one can generate an acyclic orientation by orienting each edge from smaller to larger proportion of red (with ties broken by vertex number say). Further, if  $N_1$  is the set of neighbors of vertex  $v$  with more red and  $N_2$  is the set of neighbors of  $v$  with less red, then Equation 1 simplifies to

$$df(v) = |N_1|r(v) + |N_2|b(v) + \sum_{w \in N_2} r(w) + \sum_{w \in N_1} b(w),$$

where  $b(x) = 1 - r(x)$ .

So, the proof is to enumerate the acyclic orientations. For each such orientation, we add the constraints that  $r(u) \leq r(v)$  for all arcs  $uv$ . That is, minimizing the defect for a given orientation is a linear program.

Further, if any vertex has in- and out-degree 2 for the orientation, the defect is definitely at least 2 (by Observation 3). With several pages of calculation or by using a computer, one can show that  $K_3 \square K_3$  has eight acyclic orientations (up to symmetry) that need to be considered, and then solve the eight associated linear programs. We omit the details.  $\square$

In contrast, we found a coloring of  $K_3 \square K_5$  that beats the bound of Lemma 8:

**Lemma 10**  $D(K_3 \square K_5, 2) \leq 38/13$ .

**Proof.** A 2-coloring of  $K_3 \square K_5$  is shown below. The element  $(i, j)$  of the matrix is the red-usage on vertex  $(i, j)$ .

$$\begin{bmatrix} 0 & 8/13 & 0 \\ 0 & 0 & 8/13 \\ 1 & 11/13 & 0 \\ 1 & 0 & 11/13 \\ 6/13 & 1 & 1 \end{bmatrix}$$

It can be verified that the defect of the coloring is  $38/13$ .  $\square$

The above coloring can be extended to show that Lemma 8 is not optimal for  $m = 3$  and  $n$  odd,  $n \geq 5$ , and indeed that  $D(K_3 \square K_n, 2) \leq n/2 + 11/26$  in this case. However,

this is still not best possible. For example, one can get defect  $42/11$  for  $K_3 \square K_7$  and defect  $14/3$  for  $K_3 \square K_9$  by the colorings illustrated:

$$\begin{bmatrix} 1 & 0 & 1 \\ 4/11 & 1 & 1 \\ 0 & 8/11 & 0 \\ 1 & 1 & 4/11 \\ 1/11 & 0 & 1 \\ 0 & 8/11 & 0 \\ 1 & 0 & 1/11 \end{bmatrix} \qquad \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

We used simulated annealing computer search to find upper bounds. Though we have no exact values, it seems to us that the heuristic computer search results suggest the following:

**Conjecture 3** (a) If  $m + n$  is odd, then  $D(K_m \square K_n, 2) = (m + n - 3)/2$ .  
(b) If  $mn$  is odd and greater than 9, then  $D(K_m \square K_n, 2) < \lceil m/2 \rceil + \lceil n/2 \rceil - 2$ .

Note that Conjecture 3 (a) is trivially true for the case that  $m = 2$  (or  $n = 2$ ), since  $D(K_2 \square K_n, 2) \geq D(K_n, 2) = (n - 1)/2$ .

Proposition 9 yields the following lower bounds:

**Corollary 3**

(a) If both  $m$  and  $n$  are odd,  $D(K_m \square K_n, 2) \geq (m + n)/2 - 2 + 1/(2m) + 1/(2n)$ .  
(b) If  $m$  is even and  $n$  is odd,  $D(K_m \square K_n, 2) \geq (m + n)/2 - 2 + 1/(2n)$ .

For more colors we have one trivial observation: that  $D(K_n \square G, k) = \lceil n/k \rceil - 1$  for any  $k$ -partite graph  $G$ , as a corollary of Proposition 1.

### 3.5 Regular graphs

Lovász [6] showed that we can ordinarily 2-color a cubic graph with defect at most 1. Therefore  $D(G, 2) = 1$  for all nonbipartite cubic graphs  $G$ .



For a 4-regular graph, Lovász's result shows that one can ordinarily 2-color it with defect at most 2. We conjecture that this can be improved. Proposition 8 shows that the composition  $G = C_m[2K_1]$  where  $m$  is odd has  $D(G, 2) = 2$ . Using simulated annealing (that is, a randomized search for a coloring), the computer can find a 2-coloring with defect smaller than 2 for all 4-regular graphs on up to 14 vertices, except for the compositions of odd cycles, and the two graphs  $K_5$  and  $K_3 \square K_3$ , which we saw earlier have minimum defect 2. We conjecture a general behavior:

**Conjecture 4** *Apart from  $G = C_m[2K_1]$  where  $m$  is odd, it holds that  $D(G) < 2$  for all but finitely many connected 4-regular graphs.*

## 4 Complexity

Unsurprisingly, it is NP-hard to determine if there is a coloring with defect at most some specified  $d$ .

One way to see this is that fractional defect 2-coloring is NP-hard even for  $d = 1$ . One can extend Lemma 2 to show that in graphs of minimum degree at least 3, a 2-coloring with defect 1 can only be a coloring with monochromatic vertices. Thus the fractional defect 2-coloring problem is equivalent to the ordinary defective 2-coloring problem in such graphs. The latter problem was shown to be NP-hard by Cowen [3]. (Actually, we need ordinary 1-defect coloring to be NP-hard in graphs with minimum degree at least 3. But one can transform a graph to having minimum degree at least 3 without changing the coloring property by adding, for each vertex  $v$ , a copy of  $K_4$  and joining  $v$  to one vertex of the  $K_4$ .)

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## References

- [1] I. Broere and C.M. Mynhardt. Generalized colorings of outerplanar and planar graphs. In *Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984)*, pages 151–161. Wiley, 1985.
- [2] L.J. Cowen, R.H. Cowen, and D.R. Woodall. Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency. *J. Graph Theory*, 10:187–195, 1986.
- [3] R. Cowen. Some connections between set theory and computer science. volume 713 of *Lecture Notes in Comput. Sci.*, pages 14–22. Springer, 1993.
- [4] D.W. Cranston, J. Kim, and W.B. Kinnersley. New results in  $t$ -tone coloring of graphs. *Electron. J. Combin.*, 20:Paper 17, 14, 2013.
- [5] D. D. Liu and X. Zhu. Fractional chromatic number and circular chromatic number for distance graphs with large clique size. *J. Graph Theory*, 47:129–146, 2004.
- [6] L. Lovász. On decomposition of graphs. *Studia Sci. Math. Hungar.*, 1:237–238, 1966.
- [7] P. Mihók. On vertex partition numbers of graphs. In *Graphs and Other Combinatorial Topics (Prague, 1982)*, pages 183–188. Teubner, 1983.
- [8] A. Pirnazar and D.H. Ullman. Girth and fractional chromatic number of planar graphs. *J. Graph Theory*, 39:201–217, 2002.
- [9] H. Xu. *Generalized Colorings of Graphs*. PhD Dissertation, Clemson University, May 2016.